1. Evaluate the limit $\lim_n \left(\frac{5n^2+2n+1}{3n^2+n+2}\right)$ by definition

2. Let $A \subseteq \mathbb{R}$. Suppose $\sup A = \alpha \in \mathbb{R}$

(a) Show that there exists a sequence (a_n) in A converging to α

(b) Show that there exists a monotone increasing sequence (b_n) in A converging to α.

3. Let $a>0$. Show that $\lim_n \frac{a^n}{n!} = 0$

4. Let $p \in \mathbb{N}$ and $b \in \mathbb{R}$ satisfy $0<b<1$. Show that $\lim (n^p b^n) = 0$

5. Let (x_n) be a sequence of positive real numbers.

Suppose $\lim_n \sqrt[n]{x_n} = L$, where L is a non-negative real number.

(a) if $0 \le L < 1$, show that $\lim_n x_n = 0$

(b) If $L>1$, show that (x_n) is divergent.

(c) What happens if $L=1$?

6. Let (x_n) be a sequence of positive real numbers.

Suppose $\lim_n \frac{x_{n+1}}{x_n} = L$, where L is a non-negative real number.

Show that $\lim_n \sqrt[n]{x_n} = L$

1. Want to show $\lim \left(\frac{5n^2+2n+1}{3n^2+n+2}\right) = \frac{5}{3}$

$$\left|\frac{5n^2+2n+1}{3n^2+n+2} - \frac{5}{3}\right| = \left|\frac{5n^2+2n+1-5n^2-\frac{5}{3}n-\frac{10}{3}}{3n^2+n+2}\right|$$

$$= \left|\frac{\frac{1}{3}n - \frac{7}{3}}{3n^2+n+2}\right|$$

$$\leq \left|\frac{\frac{1}{3}n}{3n^2+n+2}\right| + \left|\frac{\frac{7}{3}}{3n^2+n+2}\right|$$

$$< \frac{\frac{1}{3}n}{3n^2} + \frac{\frac{7}{3}}{n}$$

$$< \frac{1}{9n} + \frac{7}{3n} = \frac{22}{9n}$$

Let $\varepsilon > 0$

Choose $K \in \mathbb{N}$ s.t. $\frac{1}{K} < \frac{9}{22}\varepsilon$, $\Rightarrow \frac{1}{n} < \frac{9}{22}\varepsilon$

Then $\left|\frac{5n^2+2n+1}{3n^2+n+2} - \frac{5}{3}\right| < \frac{22}{9n} < \varepsilon$

2 a) Lemma 2.3.4 u is $\sup S \Leftrightarrow \forall \varepsilon > 0$, $\exists s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$

Since $\alpha = \sup A$, Set $\varepsilon = \frac{1}{n} > 0$, $\exists a_n \in A$ s.t. $\alpha - \varepsilon < a_n$

$\Rightarrow \alpha - a_n < \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\because a_n \in A$, $\Rightarrow a_n \leq \alpha \Rightarrow |\alpha - a_n| = \alpha - a_n < \frac{1}{n}$

Let $\varepsilon > 0$, A.P Choose $K \in \mathbb{N}$ s.t. $\frac{1}{K} < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon$

Then $\forall n \geq K$, $|\alpha - a_n| < \frac{1}{n} < \varepsilon$

$\therefore (a_n)$ is converging to α

b) Theorem 3.4.7 If $X=(x_n)$ is a sequence, then there is a subsequence of X is monotone

by (a), we have a monotone subsequence (b_n) of (a_n)

Theorem 3.4.2 If sequence $X = (x_n)$ of $\mathbb{R}$ converges to x,

then any subsequence $X' = (x_{n_k})$ of X also converges to x

2.b) Then (b_n) converges to α

Also (b_n) is bounded above by α.

If (b_n) is monotone increasing, done

If (b_n) is monotone decreasing

then $\lim b_n = \inf\{b_n : n \in \mathbb{N}\}$

$\alpha \geq \sup\{b_n : n \in \mathbb{N}\} \geq \inf\{b_n : n \in \mathbb{N}\} = \lim b_n = \alpha$

Then (b_n) is constant sequence.

$\Rightarrow$ (b_n) is also monotone increasing

3. Theorem 3.2.11 Let (x_n) be a sequence of positive real numbers such that $L := \lim \frac{x_{n+1}}{x_n}$ exists. If $L < 1$, then (x_n) converges and $\lim x_n = 0$

Use 3.2.11, $\frac{a^n}{n!} > 0$, and $\lim \frac{a^{n+1}}{(n+1)!} / \frac{a^n}{n!} = \lim \frac{a}{n+1} = 0 < 1$

Then $\lim\left(\frac{a^n}{n!}\right) = 0$

Pf of 3.2.11. Since $x_n > 0$ $\forall n$, $\lim \frac{x_{n+1}}{x_n} \geq 0 \Rightarrow L \geq 0$

Let $r \in \mathbb{R}$ s.t. $L < r < 1$, and let $\varepsilon = r - L > 0$

$\exists\, K_1 \in \mathbb{N}$ s.t. if $n \geq K_1$, $\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon$

$\Rightarrow \frac{x_{n+1}}{x_n} - L < \varepsilon$

$\Rightarrow \frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r$

Then if $n \geq K_1$, $0 < x_{n+1} < r x_n < r^2 x_{n-1} < \dots < r^{n-K_1+1} x_{K_1} = \underbrace{(r^{-K_1} x_{K_1})}_{C} \cdot r^{n+1}$

Since $0 < r < 1$ and $\lim(r^n) = 0$

$\Rightarrow \exists\, K_2 \in \mathbb{N}$ s.t. if $n \geq K_2$, $|r^n| < \varepsilon / C$

Let $\varepsilon > 0$, take $K = \max\{K_1, K_2\}$, if $n \geq K$

$|x_n| < C r^n < \frac{\varepsilon}{C} \cdot C = \varepsilon \Rightarrow \lim x_n = 0$

4. Use 3.2.11 since $n^P b^n > 0$, $\lim \left(\frac{(n+1)^P b^{n+1}}{n^P b^n} \right) = \lim \left(\frac{n+1}{n} \right)^P \cdot b$

$$= \lim \left(1 + \frac{1}{n} \right)^P \cdot b$$

$$= b < 1$$

Then $\lim n^P b^n = 0$

5. a) Since $0 \le L < 1$, $\exists r \in \mathbb{R}$ s.t. $L < r < 1$

Let $\varepsilon := r - L > 0$, $\exists K_1 \in \mathbb{N}$ s.t. if $n \ge K_1$

$$|\sqrt[n]{x_n} - L| < \varepsilon$$

$$\Rightarrow \sqrt[n]{x_n} - L < \varepsilon$$

$$\Rightarrow \sqrt[n]{x_n} < L + \varepsilon = r$$

$$\Rightarrow x_n < r^n$$

Since $\lim r^n = 0$, $\exists K_2 \in \mathbb{N}$ s.t. if $n \ge K_2$

Let $\varepsilon > 0$, $|r^n| < \varepsilon$

Take $K = \max\{K_1, K_2\}$

if $n \ge K$,

$$|x_n - 0| = |x_n| < |r^n| < \varepsilon.$$

Then $\lim x_n = 0$

b) Let $\varepsilon := \frac{L-1}{2}$, $\exists K \in \mathbb{N}$ s.t. if $n \ge K$

$$|\sqrt[n]{x_n} - L| < \varepsilon$$

$$\Rightarrow \sqrt[n]{x_n} - L > -\varepsilon$$

$$\Rightarrow \sqrt[n]{x_n} > L - \varepsilon = L - \frac{L-1}{2} = \frac{L+1}{2} > 1$$

$$\Rightarrow x_n > \left(\frac{L+1}{2} \right)^n$$

Assume (x_n) is convergent

$\exists M > 0$ s.t. $|x_n| \le M$ $\forall n$

since $\frac{L+1}{2} > 1$, $\frac{L+1}{2} = 1 + s$ for some $s > 0$

5b) Then $\left(\frac{L+1}{2}\right)^n = (1+s)^n = 1 + ns + \dots$

Choose $n \geq \frac{M}{s}$

Then $\left(\frac{L+1}{2}\right)^n = 1+ns+\dots \geq M+1$

Then. $M \geq |x_n| > \left(\frac{L+1}{2}\right)^n \geq M+1$.

contradiction

5c) $x_n = 1$, $\Rightarrow$ $\lim x_n^{\frac{1}{n}} = 1$ and $\lim x_n = 1$

$x_n = n$ $\Rightarrow$ $\lim x_n^{\frac{1}{n}} = 1$ and x_n diverge P.61

6. Let $\varepsilon > 0$ $\exists K_1 \in \mathbb{N}$ s.t. if $n \geq K_1$, $\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon$

$$-\varepsilon < \frac{x_{n+1}}{x_n} - L < \varepsilon$$

$$(L-\varepsilon)x_n < x_{n+1} < (L+\varepsilon)x_n$$

$$(L-\varepsilon)^{n-K_1+1} x_{K_1} < x_{n+1} < (L+\varepsilon)x_n < (L+\varepsilon)^2 x_{n-1} < \dots < (L+\varepsilon)^{n-K_1+1} x_{K_1}$$

$$\Rightarrow (L-\varepsilon)^{n-K_1} x_{K_1} < x_n < (L+\varepsilon)^{n-K_1} x_{K_1}$$

$$\Rightarrow (L-\varepsilon)\Big(\underbrace{(L-\varepsilon)^{-K_1} x_{K_1}}_{C_1}\Big)^{\frac{1}{n}} < x_n^{\frac{1}{n}} < (L+\varepsilon)\cdot\Big(\underbrace{(L+\varepsilon)^{-K_1}\cdot x_{K_1}}_{C_2}\Big)^{\frac{1}{n}}$$

$$\Rightarrow (L-\varepsilon)C_1^{\frac{1}{n}} < x_n^{\frac{1}{n}} < (L+\varepsilon)C_2^{\frac{1}{n}}$$

since $c>0$, $(c^{\frac{1}{n}})$ converges to 1. $\exists K_2 \in \mathbb{N}$, if $n \geq K_2$

$$|c^{\frac{1}{n}} - 1| < \varepsilon$$

$$1-\varepsilon < c^{\frac{1}{n}} < 1+\varepsilon$$

Take $K = \max\{K_1, K_2\}$, if $n \geq K$

Then $x_n^{\frac{1}{n}} < (L+\varepsilon)(1+\varepsilon)$

$$x_n^{\frac{1}{n}} < L + (1+L)\varepsilon + \varepsilon^2$$

$$\Rightarrow x_n^{\frac{1}{n}} - L < (1+L)\varepsilon + \varepsilon^2$$